THE EFFECT OF PRACTICE IN THE CASE OF A PURELY INTELLECTUAL FUNCTION

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The mental multiplication of one three place number by another affords a convenient means of studying several interesting psychological topics. For instance, the process affords perhaps the best brief test of attention of those so far used; the nature of the images in which one thinks is shown perhaps better by such a real mental problem than by questions concerning one's power of voluntary recall of images; the efficiency of the process is readily measurable so that it serves well as a test of fatigue or practice. It is especially advantageous for the study of practice because it requires no apparatus and offers a case of improvement in a function which a student of very slight psychological training can readily understand and measure. The experiment which is reported here might well be made as a part of the class work of a course in psychology.

I shall not rehearse all the details of the management of the experiment or all its results, but shall confine this report to the facts necessary for the understanding and criticism of certain conclusions concerning the amount, rate, progressive change of rate and spread of improvement.

THE EXPERIMENT

After preliminary training with three or four examples in mental division of a 6 place by a 2 place number, and two examples in mental multiplication of 3 place by a 3 place number, 33 individuals multiplied mentally from 50 to 96 examples like those quoted below,[1] which are a random selection in random order of the examples made by putting any 3 place number containing no digit lower than 3 and repeating no digit, with any other such 3 place number.

Of the 33 individuals 1 did only 50, 1 only 60, 1 only 66, 1 only 75, and 1 only 85 examples. The remaining 28 did 96 each. In what follows only the 28 individuals will be considered, unless a special statement to the contrary is made. As a rule 5 or 6 examples were done per day. The time of day varied amongst

[1] 657 398 479 358 589 395 396 864 739 983
964 367 476 537 745 359 953 659 459 394

individuals and in some cases within the different practice periods of the same individual. It was impossible to prevent these variations in conditions without imposing great inconvenience on the subjects. Such variations increase somewhat the variable errors of all the determinations, but if one is careful to interpret differences in results with full awareness of these differences in conditions, no serious harm need result.

Each example was done as follows. A time at which to start was set and recorded. At this time, say A. M. 8 hrs. 40 min. 30 s., the example was taken up, looked at long enough to fix the two numbers in memory so well that they could be repeated from memory and further memorized without the paper. The example was then laid aside, no sensory aids were used, and when the full answer was obtained it was written down and the time recorded when the last figure of it had been written. If the subject was interrupted *ab extra* as by a knock at the door, the record was omitted, the same example being done a day or so later. The subjects were allowed to examine their results in comparison with the correct answers.

Reduction of the Scores to One Variable

For the purposes of this article the following scores were used: (1) The times taken in doing the 1st, 2nd, 3rd, 4th, 5th, 91st, 92nd, 93rd, 94th, 95th, combined times for 1-5, 91-95, 86-90, 1-10, 11-20, 21-30, 31-40, 41-50, 51-60, 61-70, 71-80, and 81-90. (2) The errors made in each of the above examples or groups of examples, an error being defined as any wrong figure in the answer, 6 errors being the worst possible record for an example in accuracy.

When it is desirable to have a single measure of efficiency, I transmute errors into time by adding 1/10 of the time taken per example in lieu of each error made. Thus a record of 200 seconds and 1 error for an example becomes 220;—a record of 2,500 seconds and 13 errors for ten examples becomes $2{,}500+(13\times\frac{250}{10})$, or 2,825. Any such scheme of allowance can be criticised and I do not pretend that this is the best one that could be found for the present case. It is not far wrong, however. The gross figures are given in Table I so that any one who chooses may apply any other scheme for equating time and errors. It will be found, I think, that with any rational scheme the general conclusions of the study will remain as they now are.

The Amount of Improvement

The facts from which the amount of improvement is estimated are the records of the first five examples done and the first five of the last six done, taken in connection with the time of day when it differed in the two cases. By observing

Table I. — Gross Scores

Individual	Sex	First Five		Last Five		First Ten		Second Ten		Third Ten		Fourth Ten		Fifth Ten		Sixth Ten		Seventh Ten		Eighth Ten		Ninth Ten	
		Sec.	En.	Sec.	En.	Sec.	En.	Sec.	En.	Sec.	En.	Sec.	En.	Sec.	En.	Sec.	En.	Sec.	En.	Sec.	En.	Sec.	En.
1	F	1022	16	764	1	2004	20	2565	20	2285	10	2259	27	2205	14	2081	16	1826	21	1528	21	2093	23
2	F	1243	22	680	18	2298	44	2500	38	2426	35	3131	35	2864	40	2705	48	1688	38	1703	37	1709	26
3	F	2040	2	880	3	4200	9	3120	15	3305	15	2070	7	1990	14	2220	7	2058	11	2412	18	2406	2
4	M	2130	7	600	1	3390	13	2580	6	1740	8	1275	6	1050	2	1020	7	1110	2	960	4	990	7
5	F	3000	12	1230	5	6300	24	4700	32	3550	23	3240	24	3375	15	2950	16	2730	15	3300	18	2685	12
6	M	1315	5	867	3	2422	15	2433	13	2470	19	2352	12	2313	18	2112	17	2053	19	1865	14	1862	10
7	F	2185	16	1803	5	5500	22	7630	26	6877	16	8483	14	6800	11	6248	9	6519	12	4440	13	4452	7
8	F	1980	14	990	11	4440	25	1950	40	2400	25	2160	13	2670	20	1785	17	1395	23	1855	16	2080	15
9	F	1370	15	550	2	3181	27	2493	27	2100	14	1615	18	1701	13	1400	15	1327	16	1010	14	1085	6
10	F	2400	12	1230	10	3570	17	3375	6	2778	8	2445	4	2445	27	2527	11	2305	14	2167	10	2400	21
11	F	3535	12	1915	5	6442	23	5331	19	5317	17	5160	21	4135	14	4231	20	4108	18	3548	17	3656	10
12	F	2834	23	1511	10	5616	40	6437	4	4924	18	4196	18	5250	12	4289	19	3320	9	3286	16	4074	15
13	M	2765	6	377	6	4695	13	3447	15	2461	11	2165	10	1988	7	1597	11	1256	8	1109	15	947	10
14	F	900	3	525	3	2200	5	2765	8	2945	3	1895	8	1935	4	2105	8	1660	8	1275	8	1170	4
15	F	5340	0	2369	0	9180	1	9090	10	6293	11	6690	13	5387	7	4702	5	4495	5	3932	7	4480	6
16	F	1870	12	450	1	3505	22	2855	18	2285	20	2065	9	2467	10	2045	13	1470	12	1700	16	1315	13
17	F	2665	19	637	10	4535	37	3532	21	3201	19	3260	17	2295	15	2270	11	1823	14	1675	22	1720	18
18	M	3000	16	960	9	6120	28	4800	13	4080	16	4560	10	3420	21	3120	14	2760	21	1740	25	2460	20
19	F	2235	4	1080	0	3915	10	3660	8	3660	9	3225	9	2385	6	2370	5	2400	3	2265	5	1905	4
20	F	2040	15	825	3	4020	26	4105	8	3060	18	3375	10	2550	13	2355	5	2970	8	2565	21	2055	12
21	M	2530	11	725	17	4325	26	3825	31	3323	29	2655	32	2845	25	2365	23	2340	18	1960	30	2220	27
22	M	2368	2	971	0	3792	2	3553	4	3079	4	3258	6	2912	0	2845	5	3085	2	2294	2	1939	0
23	F	1590	13	950	14	3450	16	3790	17	3245	16	2365	5	2025	12	2290	9	2520	17	2050	15	2260	9
24	F	1857	2	740	12	3657	10	2801	23	2843	9	2347	15	2601	9	2503	19	2334	10	2232	11	1603	18
25	F	3010	21	1500	7	5135	36	4700	26	4200	29	3780	19	2990	26	4084	19	3460	21	2630	25	3760	22
26	M	1565	6	780	5	2710	15	2225	17	2465	20	1955	11	2145	18	2205	12	1750	5	1645	3	1870	11
27	M	2720	5	820	4	4870	13	4000	18	3270	10	2390	11	2615	10	2400	13	2430	16	2160	11	2280	15
28	M	3525	9	1022	10	6340	15	4430	22	3631	16	3174	18	2960	13	2688	16	2418	13	2394	13	2139	15
29	M	.....	....	.....	....	3750	11	3580	21	3355	21	2348	14	2166	6	.....	....	.....	....	.....	....	.....	....
30	M	.....	....	.....	....	4145	13	3281	11	2379	9	2405	10	2008	19	1550	11	1660	13	1700	12	.....	....
31	M	.....	....	.....	....	4672	7	4448	2	3693	6	2154	7	2019	2	1798	3	1428	4	.....	....	.....	....
32	M	.....	....	.....	....	13157	2	13115	12	.....	....	.....	....	.....	....	.....	....	.....	....	.....	....	.....	....
33	M	.....	....	.....	....	8965	2	6610	5	.....	....	.....	....	.....	....	.....	....	.....	....	.....	....	.....	....

the gross scores, and not only the scores as equated for errors but also the cases where the initial and final records were identical in respect to accuracy,[1] we can make a reasonable prediction concerning the reduction in time which would have occurred had the individual worked at the beginning and at the end of the practice with the same accuracy.

The ratios of such scores for the last five examples to those for the first five were as follows: 14, 20, 21, 23, 26, 28, 29, 30, 31, 32, 34, 36, 39, 42, 42, 44, 47, 48, 50, 50, 50, 50, 52, 58, 59, 60, 64, 70. The median is .42 (P. E. .02) and the median deviation from it is .10. The separate scores are subject to somewhat large variable errors so that it would be unsafe to infer much from the range of variation.

This estimate of the general amount of improvement would be very, very slightly altered by any reasonable system of equating errors and time. This can be demonstrated by actual trial of such systems and also by taking those cases where the difference in accuracy between the first and last five examples was nil or slight. For the eleven such individuals the median of the ratios of the scores of the last five examples to the corresponding ratios of the first five was .41 (P.E. .03).

The fact that these mature and competent minds improved in the course of so short a training so much as to be able to do an equal task in two-fifths of the time first taken is worthy of attention because of its bearing upon the problem of the influence of improvement in one function upon the efficiency of other functions. It is clear first that the training which this group had had for twenty odd years in remembering facts, resisting distractions and carrying in mind a complex series of relationships had left this special function of mentally multiplying a three place number by a three place number in a very easily improvable condition. Such could not have been the case if the components of that previous training had exerted each even a very moderate general influence. It is clear also that this improvement of over fifty per cent. must have been restricted closely to the special function involved. The most ardent advocate of the general influence of specific practice would not, I judge, claim that ten hours' drill in any one thing could improve an already well educated adult 50 per cent. or 5 per cent. or even 1 per cent. in the average of all his intellectual processes.

In estimating individual differences in the amount of improvement and in estimating the relations of these differences to other mental characteristics of the same individuals, the ratios listed

[1] This comparison will give only a limit, for it means for a person who improves in both speed and accuracy that a better early record than usual is compared with a worse late record than usual.

above must not be taken thoughtlessly at their face value. For a person to change from 400 seconds per example to 200 is not necessarily the same amount of improvement as for him or another to change from 200 seconds to 100 seconds. The second is probably an improvement which fewer individuals would be capable of, which the same individual would take longer to attain, and which, if analyzed into its constituents, would be found to be different from the other. To call the two equal as fractions must not lead one to infer any thoroughgoing equality in the facts which the fractions only partially represent. The relation of one-half of a man to a whole man is by no means the same as the relation of one-half of an earthworm to a whole earthworm, or of one-half of a dollar to a whole dollar. In fact, every measure of improvement by a gross difference or by a ratio must be accompanied by a statement of the initial or final gross actual ability.

It is beyond the province of this article to discuss the intricacies of methods of measuring change. The aim here is only to show and very roughly measure those differences by a method to which no one can properly object. Consider, then, the following eight records:

GROSS RECORDS

Initial (I)		Final (F)		Ratios F/I		Estimated Single F/I Ratios, errors being equated into time
Time	Errors	Time	Errors	Time	Errors	
2765	6	377	6	14	100	14
1870	2	450	1	24	8	20
2665	19	637	10	24	53	21
2130	7	600	1	28	14	26
2185	16	1803	5	82	31	70
1590	13	950	14	60	108	59
3535	12	1915	5	54	42	50
2834	23	1511	10	53	44	50

Now whether we regard a poor initial record as favorable to later improvement or not; whether we mean by twice as much improvement twice as much gross reduction in time or twice as much percentile reduction or twice as low an ending-beginning ratio—in any case we find some one of the first group who improves two and a half times as much as some one of the second group. There is, then, a range of at least two and a half to one among the 28 students on any reasonable and on

most unreasonable methods of scoring improvement. Nor would the unreliability of the measures of individual improvement be any more likely to decrease than to increase this range.

An investigation of the relationship of this difference in amount of improvement to other differences amongst the 28 individuals becomes necessarily very complex, and I shall not present the evidence here. There is a positive correlation with general intellectual achievement, a correlation which I estimate roughly as at least .4, possibly much higher.

There is apparently a zero or a slight negative relationship with the vividness and fidelity of visual images of the numbers, partial products, etc. The proportions of those of strong and of weak visual images were closely the same in those improving much and those improving little. Of the few cases who reported increase or decrease in the strength of the visual images of the numbers during the course of practice, those who reported a decrease improved somewhat more. It is also significant that more individuals reported a decrease than did an increase (9 and 3 respectively).

The Limits of Practice Effect

For the function practiced, the multiplication of a three place number by a three place number, the physiological limit is, for a capable person, very, very low, for such a person could, by devoting himself absolutely to it long enough, arrive at a knowledge of a large part of the multiplicalion table up to 999 times 999, and at an absolute knowledge of the multiplication table up to 99 times 99. There is no question of the attainment of such a final limit of practice in this experiment, but one individual (No. 4) did reach a condition beyond which the remaining practice of the experiment itself did not appreciably improve him. (See Fig. 1.) Such was possibly the case also with individual 9.

In view of the fact that the ultimate limit is far below the ability recorded by subject 4, the arrest of practice effect at this level may be taken to represent a 'plateau' from which the curve would sometime descend.

Changes in the Rate of Improvement

Practice in mental multiplication with two three place numbers is not well fitted to show changes in the rate of improvement because of the large variation in the result for any one example which a slight lapse of attention or memory may cause, though possibly it is as suitable as any equally complicated and difficult purely mental function would be. For this special purpose the presentation of the numbers themselves to sense perception, or the use of two place numbers, might be better. However, certain facts are shown with sufficient clearness and reliability.

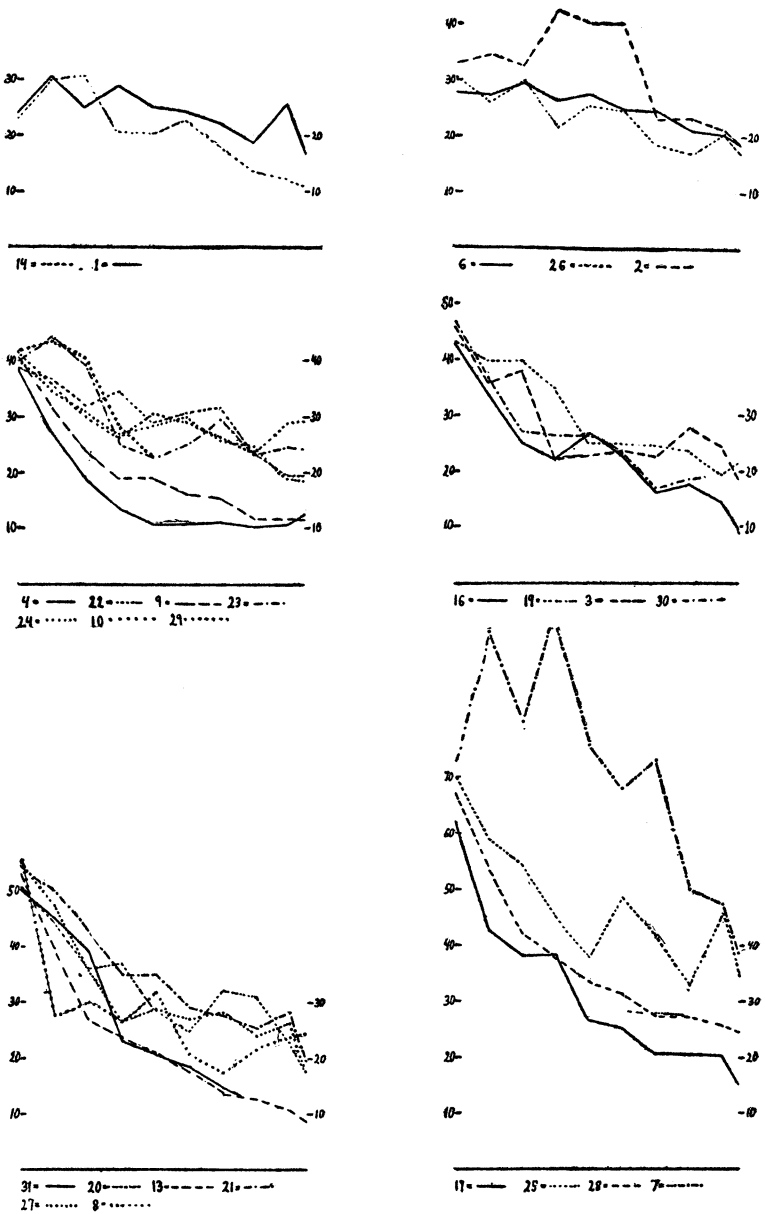

FIG. 1. The changes in the Rate of Improvement in Individuals. The course of practice runs from left to right, the whole abscissa-

length equalling 95 examples. The length of the curve represents (in hundreds of seconds) the time required to do ten examples plus the allowance for errors made. Individuals are grouped according to their degree of ability in the first ten examples. For the reasons stated on pages 375, 377 and 378 only the general sweep of each curve should be considered in arguing concerning individual differences.

In general, the earlier periods of practice show the greatest gross reduction in the scores. The graphic records of the individuals (Fig. 1) show this change.

FIG. I (Continued).

The apparent changes in the rate of improvement, that is the forms of the practice curve, are widely different amongst different individuals. This, again, is clear from the graphic records. These apparent changes in the rate of improvement are due in part to the variations in conditions from which the effect of mere practice *per se* must be freed before one can prove that the law of change in the rate of improvement varies with individuals and, if so, how far it varies.

If, however, there were one law of change of rate of improvement from the start through this period of practice identical for all the 28 individuals, we could, from the present data, ascertain fairly closely what the law was. We could, that is, answer the following question:

Considering the 28 individuals as all starting at "the ability given by 10 to 20 years of general experience with mental work," and ending with 'the ability given by 10-20 years of general mental work plus the mental multiplication of 95 examples,' and considering the change in their rates of improvement from the start to the end to be due to one general law of change of rate plus individual deviations from it due to internal and external disturbing factors, what is this general law of change of rate?

The answer to this question is given by the continuous line of Fig. 2, which presents approximately the one rate of change from which the 28 separate rates of change could come with the least improbability as a result of disturbing causes. It is obtained by eliminating the total amount of change from consideration in every case by taking the *differences*—score for examples 1 to 10 minus score for examples 11 to 20, and so on up to 81 to 90, and dividing them by the total change, *i. e.*, score for 1 to 10 minus score for 81 to 90. We have, then, 28 practice curves all beginning at 100 and ending at 0, and can find the one such curve which represents the central tendency of them all.

It might well be that though no such one law held for the change of rate of improvement from the beginning to the end of the practice given in such an experiment, some one law

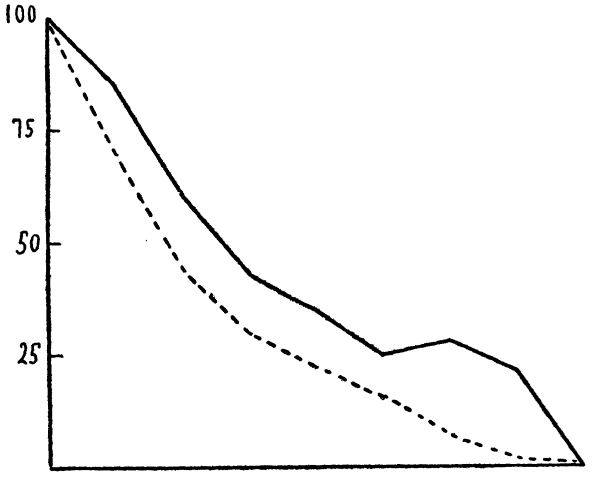

FIG. 2. The General Law of Change of Rate of Improvement in the Case of each of Two Suppositions.

might hold for this change of rate of improvement from a given ability (say to do 10 three place examples in 50 minutes

with 15 errors) to another given ability (say to do them in 25 minutes with 10 errors). That is, there might be identity in the rate of change in improvement amongst individuals whose total improvement was identical, so that significance attaches to the answer to the following hypothetical question:

Considering the changes in the rate of improvement from any one given degree of ability to any other given degree of ability to be due to one general law of change of rate plus individual deviations from it due to internal and external disturbing factors, what is this general law of change of rate?

This general law for the case of progress from a score of 4,000 to a score of 2,000 for ten examples is approximately that given by the dotted line of Fig. 2. It is obtained by taking the individuals[1] who, in some succession of tens of their practice, progressed from 4,000 or more to 2,000 or better, and plotting for each a curve irrespective of the *amount* of practice that carried them from a 4,000 score to a 2,000 score. The curves, that is, all start at 4,000, all end at 2,000, and all occupy the same length of the abscissa, so that they vary in the one element of the rate of change of improvement. The dotted line of Fig. 2 represents the one such curve from which the separate curves could be derived with the least improbability.

The reader will understand that the writer does not attempt to decide whether there is, for either case, any such one general law. As was stated on a previous page, three place mental multiplication is not a specially favorable case to study the issue and the measurement of the influence of external factors could not, in the present study, be made satisfactorily. So far as the evidence does go, it favors the conclusion that the differences amongst individuals in the changes in rate of improvement are due not only to the influence of one same law plus differences in conditions, but also to the action of radically different laws acting on different individuals according to the different physiological changes in them to which the improvement is due. The curves of Fig. 2 would then be mongrels representing no significant laws of nature.

The Influence of Equal Practice upon Individual Differences

Experiments in practice offer evidence concerning the relative importance of original nature and training in determining achievement. In so far as the differences amongst individuals in the ability at the start of the experiment are due to differences of training, they should be reduced by further training given in equal measure to all the individuals. If, on the

[1] In this case two individuals not in the 28 were included since the completion of the entire 96 examples is here irrelevant.

contrary, in spite of equal training the differences amongst individuals remain as large as ever, they are to be attributed to differences in original capacity.

As a matter of fact in this experiment the larger individual differences *increase* with equal training, showing a positive correlation of high initial ability with ability to profit by training. The data are given in Table II.

TABLE II

The ratios of the worse to the better records, early and late in the course of practice.

The numbers 1 to 28 refer to the records in order of excellence, the same number thus possibly meaning different individuals.

Records Compared	For First 5 Examples	For First 10 Examples	For Second 10 Examples	For Ninth 10 Examples	For Eighth 10 Examples	For Last 5 Examples	Relation of late to early variability, by different measures of it
21/8	1.73	1.68	1.61	1.44	1.42	1.41	Less
22/7	1.93	1.88	1.71	1.57	1.49	1.60	Less
23/6	2.21	2.04	1.92	1.78	1.59	1.95	Less
24/5	2.32	2.36	2.08	2.25	2.70	2.81	Greater
25/4	2.44	2.59	2.31	2.84	3.76	3.18	Greater
26/3	3.00	2.83	2.44	3.27	4.07	3.48	Greater
27/2	3.36	3.29	3.53	3.66	4.48	4.58	Greater
28/1	5.60	4.01	3.85	5.02	4.58	5.61	Greater

It is impossible as yet to demonstrate how far this influence of equal practice extends amongst the important mental functions, partly because common life does not make the experiment of equal practice often enough for us, and partly because comparable units for the measurement of mental achievement are so often lacking. To the author the achievements of students in schools and colleges seem to show in general that the greater original capacity gains as much or more from the same environmental training, and the differences amongst individuals who have all been brought practically to their physiological limit in the case of speed of reading, musical technique, ability in science or business or the like seem to be in general greater than the differences amongst the same individuals at earlier equivalent stages in practice. Moreover, it seems extremely probable from many facts of dynamic psychology that the man who has the capacity to improve to a given small degree more quickly than another should also improve more quickly to the next degree and should also, by and by, be capable of improving to a higher degree if given the maximum of efficient training.